Sigma (Summation notation) is used to write math. expressions more compactly

→ $\sum_{K=1}^{5} K = 1+2+3+4+5$

1 is the lower limit of summation,
5 is the upper limit of summation.

→ $\sum_{K=1}^{125} \frac{1}{K^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \frac{1}{125^2}$.

→ Let $a_1, a_2 \ldots a_n$ be real numbers and n be a positive integer. Then.

$$\sum_{K=1}^{n} a_K = a_1 + a_2 + a_3 + \cdots + a_n ,$$

K is called the index of summation.

$$\sum_{K=0}^{\infty} a_K = a_0 + a_1 + a_2 + \cdots a_n + \cdots$$

↑ infinite sum.

Riemann Sums: Left and Right Sums

Let f be a continuous, nonnegative ($f \geq 0$), increasing function on the interval $[a,b]$.

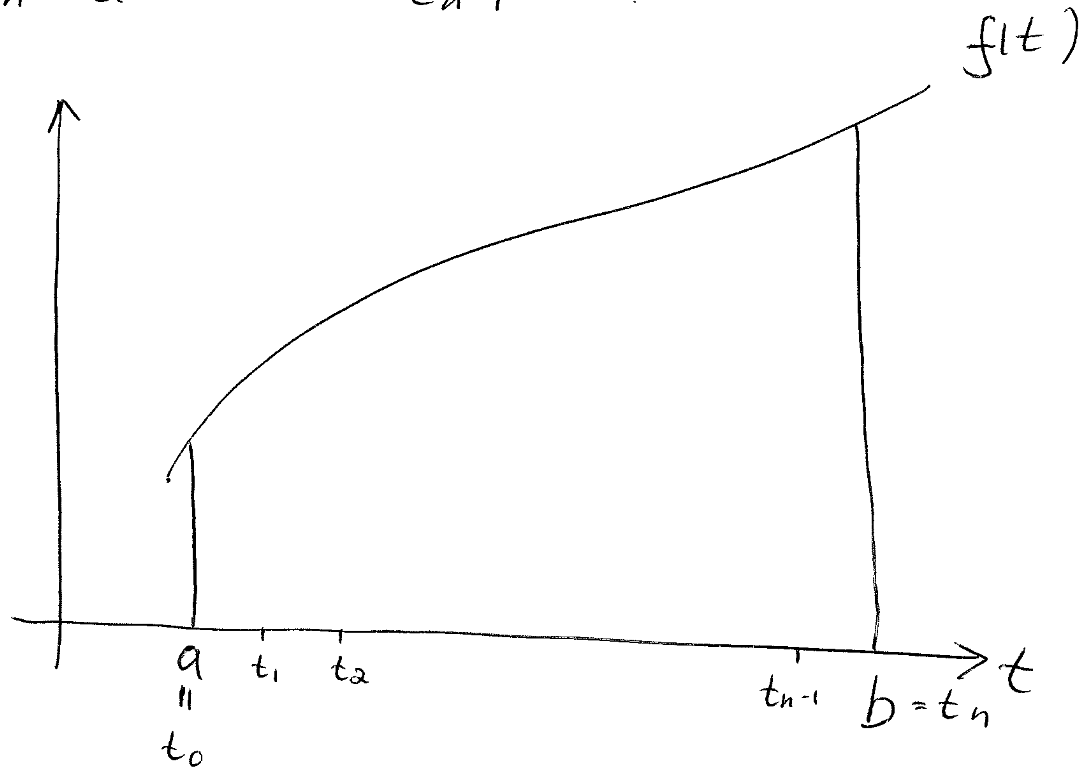
We partition $[a,b]$ into n subintervals of equal length : $\frac{b-a}{n} = \Delta t$

$t_1 = a + \Delta t \; = t_0 + \Delta t$

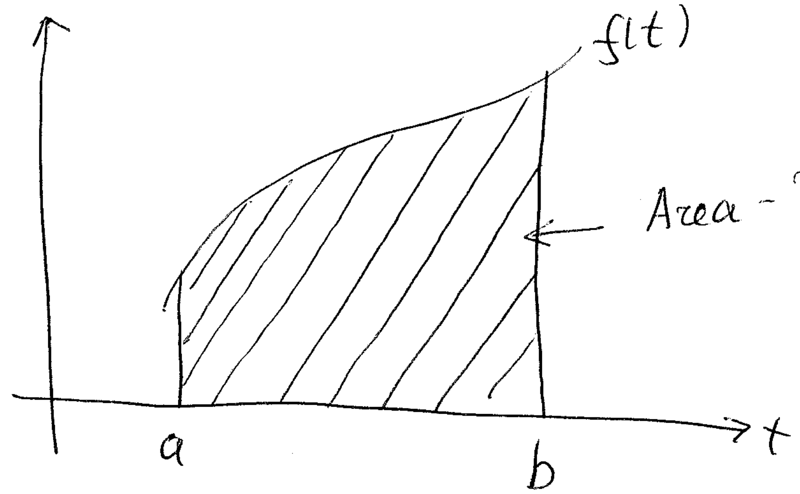
$t_2 = a + 2\Delta t = t_0 + 2\Delta t = t_1 + \Delta t$

$\vdots$

$t_n = a + n\Delta t = t_{n-1} + \Delta t$

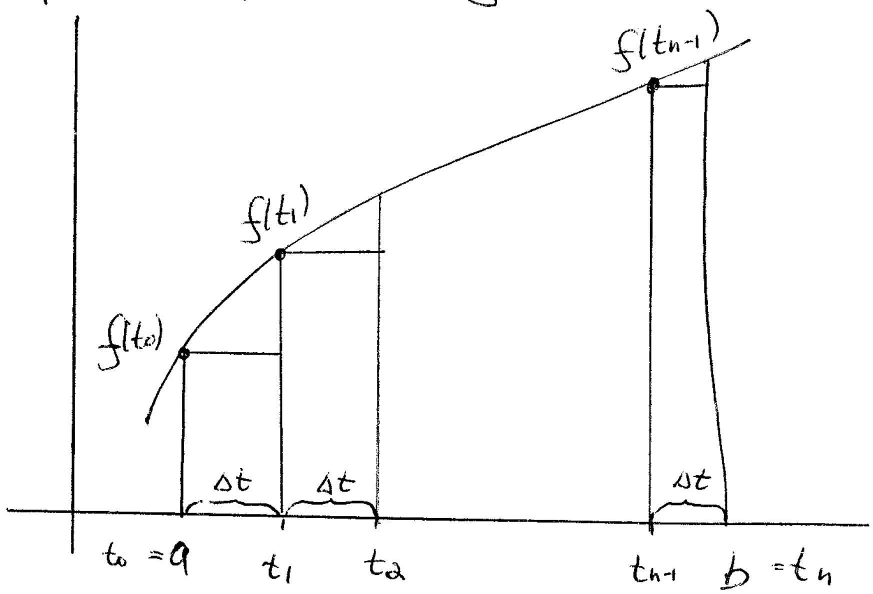


Our Goal: We want to find the area of the region between the graph of f, and the t-axis from a to b.

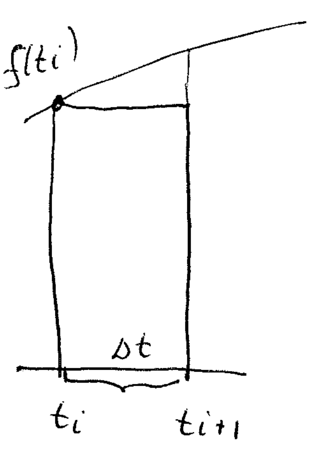


Consider the ~~the~~ following two cases:

$\boxed{\text{Case 1}}$ We take measurements of $f(t)$ at the left end point of each subinterval, and



build rectangles that have width Δt and heigth $f(t_i)$, i changes from 0 to $(n-1)$



The area of the first rectangle is $f(t_0) \cdot \Delta t$.

The area of the second rectangle is $f(t_1) \cdot \Delta t$. and so on.

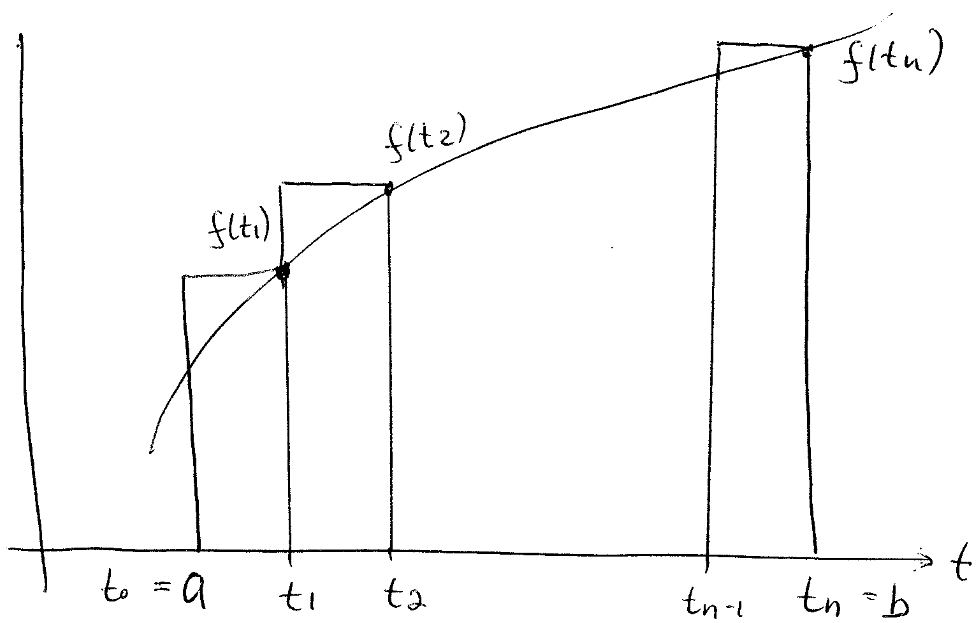
The area of the first stripe is $\approx f(t_0)\cdot\Delta t$,
the area of the second stripe is $\approx f(t_1)\cdot\Delta t$ and so on.

The sum of $f(t_0)\cdot\Delta t$, $f(t_1)\Delta t$, ... $f(t_{n-1})\cdot\Delta t$ gives us an approximate value of area of the region.

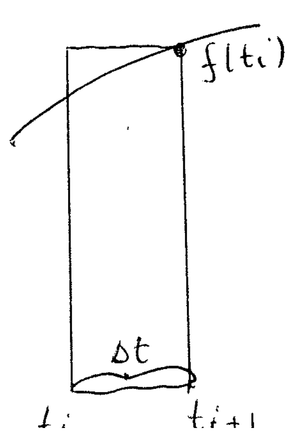
$$f(t_0)\cdot\Delta t + f(t_1)\Delta t + \cdots f(t_{n-1})\Delta t = \sum_{i=0}^{n-1} f(t_i)\Delta t =$$

denote it by I_L (left hand sum)

Case 2 Now we take measurement of $f(t)$ at the right end point of each subinterval.



We buidt rectangles that have width Δt and height $f(t_i)$.



The area of the first rectangle is $f(t_1)\cdot\Delta t$

The area of the second rectangle is $f(t_2)\cdot\Delta t$ and so on.

Summing up the areas, we obtain an approximate value of the area of the region:

$$f(t_1)\cdot\Delta t + f(t_2)\Delta t + \cdots \quad f(t_n)\cdot\Delta t =$$

$$= \left(f(t_1) + f(t_2) + \cdots \quad f(t_n)\right)\Delta t =$$

$$= \sum_{i=1}^{n} f(t_i)\Delta t \underset{\text{denote by}}{=} I_R \quad (\text{the right hand Riemann sum})$$

As the width Δt of the rectangles approaches zero ($\frac{a-b}{n} = \Delta t \to 0,\ n \to \infty$), the rectangles fit the curve of the graph more exactly and the sum of their areas gets closer and closer to the area under the curve.

$$\boxed{I_L \le \underset{\text{region}}{\text{Area of the}} \le I_R}$$

$\boxed{\text{Remark 1}}$ If f is an <u>increasing</u> f-n then I_L increases and I_R decreases as $n \to \infty$.

$\boxed{\text{Remark 2}}$ If f is a <u>decreasing</u> f-n then

$$I_R \le \underset{\text{region}}{\text{Area of the}} \le I_L$$

and I_R increases and I_L decreases as $n \to \infty$.

→ Let n go to infinity:

Thus, the area of the region between the graph of $f(t)$ and the t-axis from a to b is approximately equal to $\lim_{n\to\infty} I_L = \lim_{n\to\infty} I_R =$

= some number that we call the definite integral of $f(t)$ from a to b and denote by $\int_a^b f(t)dt$.

In other words, $\lim_{n\to\infty} I_L = \lim_{n\to\infty} I_R = \int_a^b f(t)dt =$

= area of the shaded region = $\boxed{\text{some number}}$

In the general case,

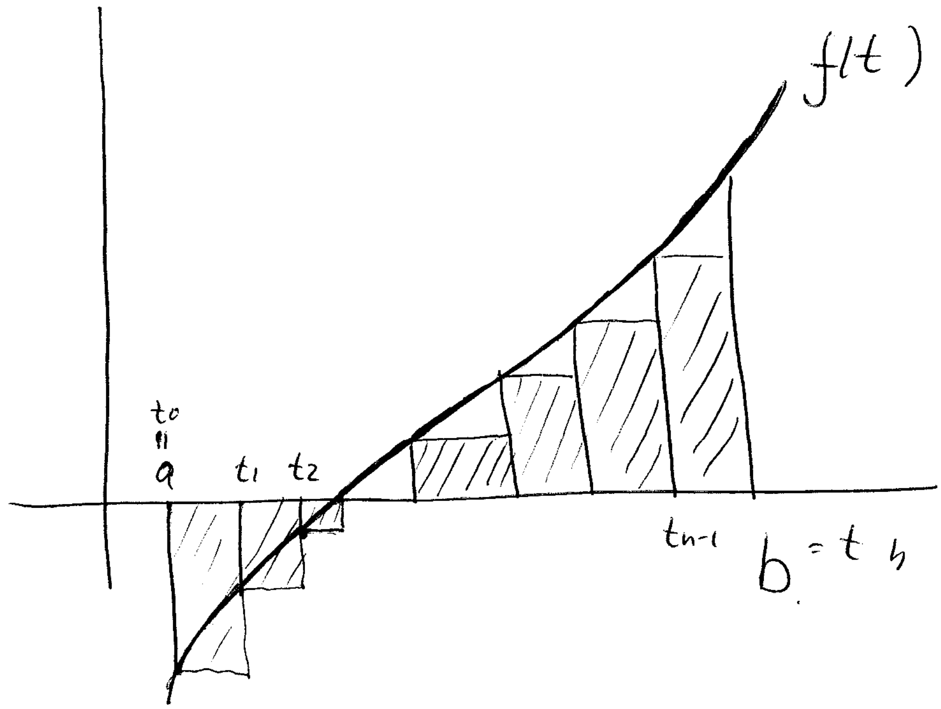
Suppose f is $\boxed{\text{continuous}}$ for $a \le t \le b$.

The definite integral of f from a to b, written $\int_a^b f(t)dt$ is the limit of the left-hand (I_L) and the right-hand (I_R) sums with n subintervals as n gets arbitrarily large (n tends to infinity):

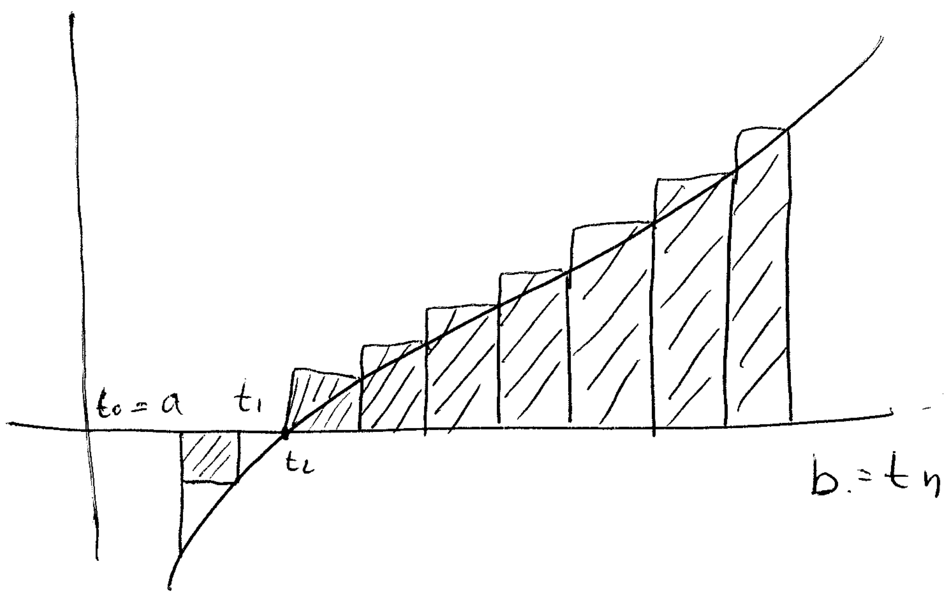
$$\int_a^b f(t)dt = \lim_{n\to\infty} I_L = \lim_{n\to\infty} \left(\sum_{i=0}^{n-1} f(t_i)\Delta t\right) \text{ and}$$

$$\int_a^b f(t)dt = \lim_{n\to\infty} I_R = \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(t_i)\Delta t\right),$$

f is called the integrand, and a and b are called the limits of integration

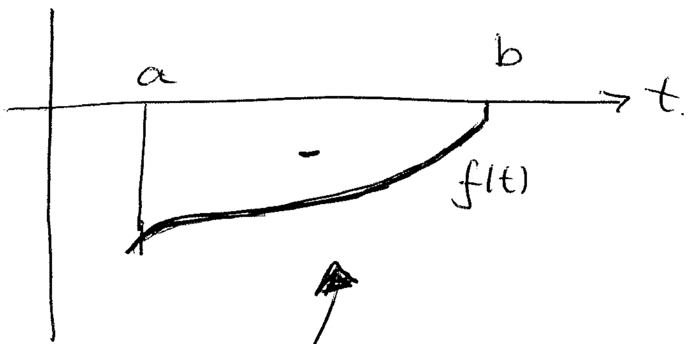


I_L



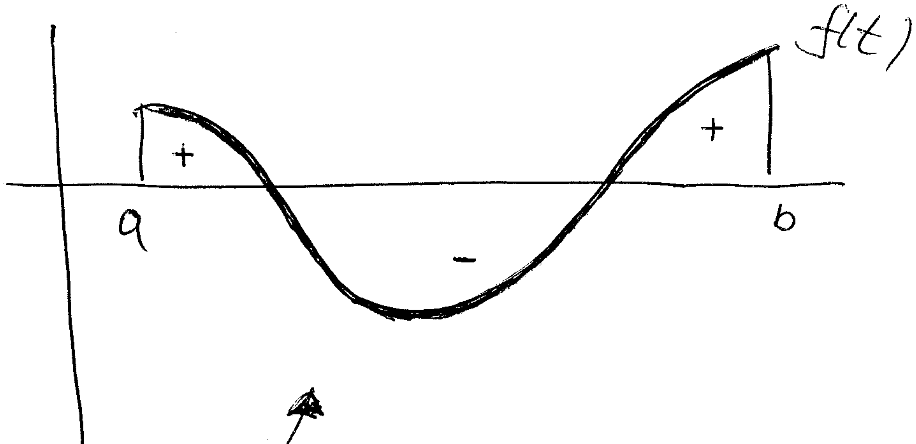
I_R

→ When $f(x)$ is not Positive. What is the meaning of $\int_a^b f(t)dt$?



$f(t) < 0$.

If the graph lies below the t-axis, then each value of $f(t)$ is negative, so each term $f(t_i)\Delta t$ is negative aswell; and the area gets counted negatively. In this case, the definite integral $\int_a^b f(t)dt$ = negative of the area



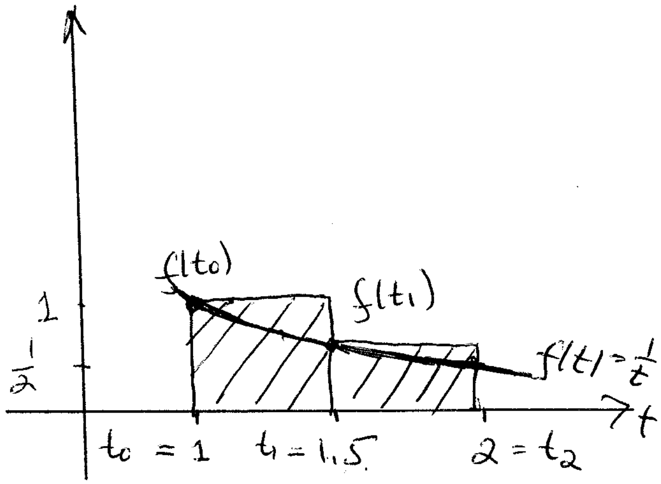
If $f(t)$ is positive for some t values and negative for other and $a < b$.

$\int_a^b f(t)dt$ = [the sum of areas above the t-axis] − [the sum of areas below the t-axis]

(Can be shown by using the Riemann Series approach).

<u>Example</u> Calculate the left-hand and the right-hand sums with $n=2$ for $\int_1^2 f(t)\,dt = \int_1^2 \frac{1}{t}\,dt$

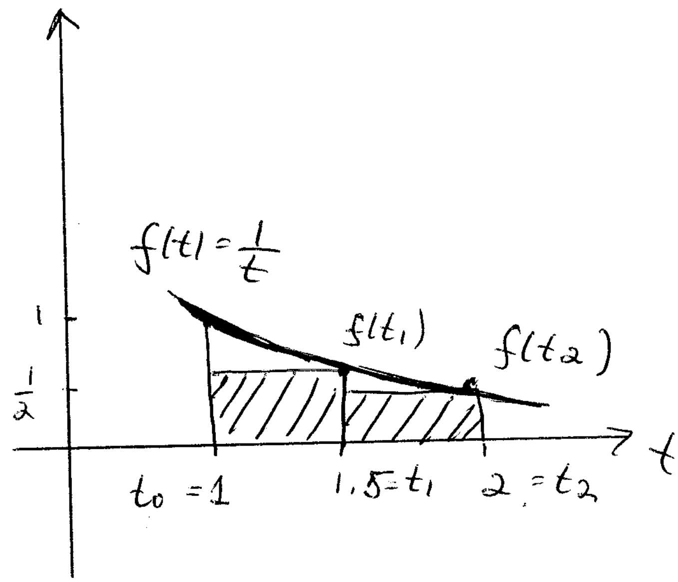
The Left-hand Sum



$a=1,\ b=2,\ n=2$

$\Delta t = \frac{b-a}{n} = \frac{2-1}{2} = \frac{1}{2} = 0.5$

The Right-hand Sum



$\Delta t = 0.5$

$f = \frac{1}{t}$ is a <u>decreasing</u> function

$t_0=1,\ f(t_0) = \frac{1}{t}\Big|_{t_0=1} = 1$

$t_1=1.5\quad f(t_1) = \frac{1}{t}\Big|_{t_1=1.5} = \frac{2}{3}$

$I_L = \sum_{i=0}^{n-1} f(t_i)\Delta t = \sum_{i=0}^{1} f(t_i)\Delta t =$

$= f(t_0)\cdot\Delta t + f(t_1)\Delta t =$

$1\cdot 0.5 + \frac{2}{3}\cdot 0.5 = 0.8333$

$t_1=1.5\quad f(t_1)=\frac{2}{3}$

$t_2=2\quad f(t_2)=\frac{1}{2}$

$I_R = \sum_{i=1}^{n} f(t_i)\Delta t = f(t_1)\Delta t +$

$+ f(t_2)\Delta t = \frac{2}{3}\cdot 0.5 + \frac{1}{2}\cdot 0.5 =$

$= 0.5833$

$$0.5833 = I_R < \int_1^2 \frac{dt}{t} < I_L = 0.8333$$

$\boxed{n=10}$ $\quad I_R = 0.6688,\quad I_L = 0.7188$

The actual value of $\int_1^2 \frac{dt}{t} = \ln|t|\Big|_1^2 = \ln 2 \approx 0.6931$

<u>Remark</u> → If $f(t) = v(t)$, where $v(t)$ is nonnegative velocity, then $\int_a^b f(t)\,dt$ is the <u>total distance traveled</u> from time $t = a$ ~~to~~ till $t = b$.

→ If $f(t) = v(t)$ ← velocity, where $v(t)$ is sometimes negative, then $\int_a^b f(t)\,dt$ represents <u>change in position</u>, rather than distance.

→ If $f(x) = \rho(x)$, where $\rho(x)$ is (positive) density of an object of length $b-a$. Then $\int_a^b \rho(x)\,dx$ represents a mass of the object.

Theorem 1 : (Properties of Limits of Integration)

If a, b, c are any numbers $a \le c \le b$ and f is a continuous f-n, then the following is true:

(a) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

(b) $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

Theorem 2 :

Let f and g be continuous functions and let C be a constant then

(a) $$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

(b) $$\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx.$$

Suppose we have two functions $f(t)$ and $F(t)$, where $F'(t) = f(t)$. In other words

$f(t)$ is the derivative of $F(t)$

$F(t)$ is called an antiderivative of $f(t)$

The derivative of function is always unique; and there are infinitely many antiderivatives of function.

→ The indefinite integral $\int f(t)\,dt$ represents a family of all antiderivatives of $f(t)$

$$\int f(t)\,dt = F(t) + C$$

→ The definite integral $\int_a^b f(t)\,dt$ is the limit of Riemann sums

$$\int_a^b f(t)\,dt = \text{a number}$$ (the distance traveled, the area under the graph and so on)

The Fundamental Theorem of Calculus (FTC) connects the two notions (the definite and the indefinite integrals)

$\boxed{\text{FTC}}$

If $F'(t) = f(t)$ is continuous on $[a,b]$ then the following is true:

$$\int_a^b f(t)dt = \int_a^b F'(t)dt \overset{\text{FTC}}{=} \int f(t)dt \Big|_{t=a}^{t=b} =$$

$$= \left(F(t) + C\right)\Big|_{t=a}^{t=b} = \left(F(b)+C\right) - \left(F(a)+C\right) =$$

$$= F(b) + C - F(a) - C = F(b) - F(a).$$

[Thus, the definite integral of the rate of change of some quantity ($\int_a^b \frac{dF}{dt}dt$) gives the total change in the quantity between $t=a$ and $t=b$ ($F(b) - F(a)$).] !

> If $F(t)$ is a height of a tree at time t, then $F'(t)$ is the rate of change of its height and $F(b)-F(a)$ is the total change of the tree height between times $t=a$ and $t=b$.

→ If $F(t)$ represents the number of infected individuals at time t, then $F'(t)$ is the rate of change of number of infected and $F(b) - F(a)$ = total change in number of infected between $t=a$ and $t=b$

→ If $F(t)$ is a position of an object at time t, then $F'(t)$ is the rate of change of the position or velocity. Then $F(b) - F(a)$ is a displacement.

$\boxed{\text{Remark}}$ Knowing the total change of some quantity, and the initial value of the quantity $(F(a))$ we are able to find the value of the quantity at time $t=b$

$$F(b) = F(a) + \int_a^b F'(t)\,dt.$$

$$\int_5^7 (e^t + \cos t - 4t)\,dt =$$

$$\int_5^7 e^t dt + \int_5^7 \cos t\,dt - 4\int_5^7 t\,dt \overset{FTC}{=}$$

$$= \left[\int e^t dt + \int \cos t\,dt - 4\int t\,dt\right]\Bigg|_{t=5}^{t=7} =$$

$$= \left[e^t + \sin t - \frac{4t^2}{2} + c\right]\Bigg|_{t=5}^{t=7} =$$

$$(e^7 + \sin(7) - 2\cdot 49) - (e^5 + \sin(5) - 2\cdot 25) =$$

$$= e^7 - e^5 + \sin(7) - \sin(5) - 48.$$

Example Let $F(t)$ be the number of bacteria (in millions) at time t.
$F(0)=5$ (Initially, we had $5\cdot 10^6$ ind.)
The rate of change of bacteria is $3t^2+5t$, i.e $\frac{dF}{dt}=3t^2+5t$.

Find (a) the total increase in the bacteria population during the 1-st hour ($F(1)-F(0)$)
(b) the population at $t=1$ (i.e $F(1)$)

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a) Using the FTC, we have that the total change in number of bacteria between $t=0$ and $t=1$ is given by

$$\int_0^1 \frac{dF}{dt}\,dt = \int_0^1 (3t^2+5t)\,dt \overset{FTC}{=} \int (3t^2+5t)\,dt \Big|_{t=0}^{t=1} =$$

$$= \left(F(t)+c\right)\Big|_{t=0}^{t=1} = \left(t^3+\frac{5t^2}{2}+c\right)\Big|_{t=0}^{t=1} =$$

$$= \left(1+\frac{5}{2}+c\right) - (0+0+c) = 1+\frac{5}{2} = 3.5 = F(1)-F(0)$$

(b) $$F(1)-F(0) = \int_0^1 \frac{dF}{dt}\,dt = \int_0^1 (3t^2+5t)\,dt \Rightarrow$$

$$F(1) = F(0)+\int_0^1 (3t^2+5t)\,dt = 5+3.5 = 8.5$$

$F(1)$: the population at $t=1$